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Stability of the Difference Scheme for the Equation of Rahmatullin

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ABSTRACT

The energy estimation for the equation of Rahmatullin is derived using energy inequalities. The difference scheme of the equation of Rahmatullin is constructed and the stability of the constructed scheme is obtained.

INTRODUCTION

The problems of the fluid and gas motion in porous and granulated medium have been studied theoretically and numerically. The filtration in porous medium is the basis of many processes in chemical technology such as stream in chemical reactors with immovable layer of catalyst, flow of gas in blast furnace and shaft furnace, etc.

The classical Darcy equation does not qualitatively describe many physical processes [1-5]. For this reason derivation of equations that imply the Darcy law as a special case under the appropriate simplifying assumptions are very important. In this direction it should be pointed out investigations [3, 6, 7-12].

The equations the present paper deals with to describe the filtrations of gas and liquid are close in the ideological plan to papers [3, 8] in which multi-fluid model were used.

The aim of the present paper is to derive energy estimation for a model problem, and construct the implicit scheme, and show energy estimation similarly to the filtration equation.

The filtration equation of Rahmatullin can be obtained from the multi-fluid model equations [8, 13, 14] and by assuming the movement of a phase is non deformable.

The Equation of Rahmatullin is considered in the coordinate form:

$$\rho\frac{\partial u_j}{\partial t}+\sum_{k=1}^{2}\rho u_k\frac{\partial u_j}{\partial x_k}=-f\frac{\partial p}{\partial x_j}+\mu\sum_{k=1}^{2}\left(\frac{1}{3}\delta_j^k+1\right)\frac{\partial}{\partial x_k}\left(f\frac{\partial u_j}{\partial x_k}\right)$$
$$+\mu\sum_{k=1}^{2}\left(1-\frac{5}{3}\delta_j^k\right)\frac{\partial}{\partial x_k}\left(f\frac{\partial u_{3-j}}{\partial x_{3-k}}\right)-Ku_j+\rho g_{j,} \tag{1}$$

$$\sum_{k=1}^{2}\frac{\partial f u_k}{\partial x_k}=0. \tag{2}$$

Here u_j is the jth component of the flow velocity, p is the pressure, f is the volumetric concentration, δ_j^k is Kronecker delta, μ is the viscosity of the liquid, K is the factor of interaction $\left(K=\alpha\mu(1-f)^2/(d^2f^2)\right)$, g_j is the jth component of the external force.

Note that if $f\to 1$ then equations (1) and (2) are reduced to the Navier-Stokes equations for incompressible liquid. Moreover equations (1), (2) are extensions of equations describing flow through porous medium. After the appropriate simplifications the Darcy equation follows from (1).

We'll derive energy estimation for the model problem by using energy inequalities, and construct difference scheme, and derive difference analogue of energy estimation.

THE A PRIORI ESTIMATES FOR CONTINUAL PROBLEM

Consider the estimations for the solutions of equations (1) and (2)

$$u_j\big|_s=0, \qquad u_j\big|_{t=0}=a(x) \tag{3}$$

Consider also the equations (1), (2) in the area $Q_T=\Omega\times(0,T)$, where Ω is the rectangle with boundary S $(0\le t\le T)$.

To derive an estimation we multiply the equation (1) by scalar u_j and integrate over Ω. We transform expressions separately,

$$\int_\Omega \rho \frac{\partial u_j}{\partial t} u_j dx = \frac{1}{2}\int_\Omega \rho \frac{\partial u_j^2}{\partial t} dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho u_j^2 dx, \tag{4}$$

As a next step, transform the convective term. One arrives at:

$$\begin{aligned}
&\int_\Omega \rho \sum_{k=1}^{2} u_k \frac{\partial u_j}{\partial x_k} u_j dx = -\int_\Omega \left[u_j \frac{\partial}{\partial x_1}\left(\rho u_1 u_j\right) + u_j \frac{\partial}{\partial x_2}\left(\rho u_2 u_j\right)\right] dx \\
&= -\int_\Omega u_j^2 \left[\frac{\partial \rho u_1}{\partial x_1} + \frac{\partial \rho u_2}{\partial x_2}\right] dx - \int_\Omega u_j \rho u_1 \frac{\partial u_j}{\partial x_1} dx - \int_\Omega u_j \rho u_2 \frac{\partial u_j}{\partial x_2} dx \\
&= -\int_\Omega \rho \sum_{k=1}^{2} u_k \frac{\partial u_j}{\partial x_k} \cdot u_j dx.
\end{aligned}$$

Thus,

$$\int_\Omega \rho \sum_{k=1}^{2} u_k \frac{\partial u_j}{\partial x_k} \cdot u_j dx = 0.$$

If we apply the integration by parts to other terms of the equation (4), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u_j^2 dx - \int_\Omega p \frac{\partial f u_j}{\partial x_j} dx + \mu \int_\Omega \sum_{k=1}^{2}\left(\frac{\delta_j^k}{3} + 1\right) f \left(\frac{\partial u_j}{\partial x_k}\right)^2 \\
&+ \mu \int_\Omega \sum_{k=1}^{2}\left[\left(1 - \frac{5}{3}\delta_j^k\right) f \frac{\partial u_{3-j}}{\partial x_{3-k}} \cdot \frac{\partial u_j}{\partial x_k}\right] dx + \int_\Omega K u_j^2 dx = \int_\Omega \rho g_j u_j dx.
\end{aligned} \tag{5}$$

Summing up (5) over j we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho \sum_{j=1}^{2} u_j^2 dx - \int_\Omega p \sum_{j=1}^{2} \frac{\partial f u_j}{\partial x_j} dx + \mu J(v) \\
&+ \int_\Omega K \sum_{j=1}^{2} u_j^2 dx = \sum_{j=1}^{2} \int_\Omega \rho g_j u_j dx
\end{aligned} \tag{6}$$

where

$$J(v)=\int_{\Omega}\left[f\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right)^2+\frac{4}{3}f\left(\left(\frac{\partial u_1}{\partial x_1}\right)^2+\left(\frac{\partial u_2}{\partial x_2}\right)^2-\frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}\right)\right]dx.$$

In equation (6) the term containing pressure vanishes by continuity equation. We introduce the inner product

$$\left(v,w\right)_{\rho}=\int_{\Omega}\rho vw dx=\int_{\Omega}\rho(v_1w_1+v_2w_2)dx.$$

and the norm

$$\|v\|_{\rho}^2=\left(v,v\right)_{\rho}$$

in $L_2(\Omega)$.

Then (6) can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\|v\|_{\rho}^2+\mu J(v)+\bar{\alpha}\|v\|_{\bar{f}}^2=(v,G)_{\rho} \tag{7}$$

where $\bar{\alpha}=\dfrac{\alpha\mu}{d^2}$, $\bar{f}=\dfrac{(1-f)^2}{f^2}$, G is the vector of external forces.

As $\mu>0,\ \ J(v)>0,\ \ \bar{\alpha}>0$, we get from (7) that is

$$\|v\|_{\rho}\frac{d}{dt}\|v\|_{\rho}\le(v,G)_{\rho}. \tag{8}$$

If we apply the Cauchy-Schwartz inequality to the right-hand side of the inequality (8) we obtain

$$\frac{d}{dt}\|v\|_{\rho}\le\|G\|_{\rho}$$

and integrating this inequality from 0 to t, we obtain

$$\|v(x,t)\|_{\rho}\le\|v(x,0)\|_{\rho}+\int_0^t\|G\|_{\rho}dt \tag{9}$$

Integrating the equality (6) with respect to t and applying the Cauchy-Schwartz inequality to the right-hand side of it, and using the inequality (9), we obtain

$$\|v(x,t)\|_{\rho}^{2}-\|v(x,0)\|_{\rho}^{2}+2\int_{0}^{t}\mu J(v)dt+2\bar{\alpha}\int_{0}^{t}\|v\|_{\bar{f}}^{2}\leq 2\int_{0}^{t}\ \|v\|_{\rho}\|G\|_{\rho}\,dt\leq$$

$$\leq 2\int_{0}^{t}\|G\|_{\rho}\,dt\left(\|v(x,0)\|_{\rho}+\int_{0}^{t}\|G\|_{\rho}\,dt\right)\leq\|v(x,0)\|_{\rho}^{2}+\left(\int_{0}^{t}\|G\|_{\rho}\,dt\right)^{2}+2\left(\int_{0}^{t}\|G\|_{\rho}\,dt\right)^{2}$$

Here we used the inequality $2ab\leq a^{2}+b^{2}$, which allows us to write.
Thus,

$$\|v(x,t)\|_{\rho}^{2}+2\int_{0}^{t}\mu J(v)dt+2\bar{\alpha}\int_{0}^{t}\|v\|_{\bar{f}}^{2}\leq 2\|v(x,0)\|_{\rho}^{2}+3\left(\int_{0}^{t}\ \|G\|_{\rho}\,dt\right)^{2}\qquad(10)$$

THE IMPLICIT SCHEME

By following Ladyzhenskaya [15], we construct a difference scheme,

$$\rho\left(u_{j}\right)_{\bar{t}}+\frac{1}{2}\sum_{k=1}^{2}\rho\overset{-0,+k}{u_{k}}\left(u_{j}\right)_{x_{k}}+\frac{1}{2}\sum_{k=1}^{2}\rho\overset{-0}{u_{k}}\left(u_{j}\right)_{\bar{x}_{k}}$$

$$=-fp_{\bar{x}_{j}}+\frac{\mu}{2}\sum_{k=1}^{2}\left(\frac{1}{3}\delta_{j}^{k}+1\right)\left[\left(f\left(u_{j}\right)_{x_{k}}\right)_{\bar{x}_{k}}+\left(f\left(u_{j}\right)_{\bar{x}_{k}}\right)_{x_{k}}\right]\qquad(11)$$

$$+\frac{1}{2}\mu\sum_{k=1}^{2}\left(1-\frac{5}{3}\delta_{j}^{k}\right)\left[\left(f\left(u_{3-j}\right)_{\bar{x}_{3-k}}\right)_{x_{k}}+\left(f\left(u_{3-j}\right)_{x_{3-k}}\right)_{\bar{x}_{k}}\right]-Ku_{j}+\rho g_{j};\quad j=1,2$$

$$\sum_{k=1}^{2}\left(fu_{k}\right)_{x_{k}}=0.\qquad(12)$$

Notations in (11) and (12) are the same as [15]. The system (11) represents the linear system of equations.

We assume that the equations (11) is satisfied at inner points, and (12) is in addition satisfied on the left and bottom of boundary.

To obtain estimate we multiply (11), taken in l th time layer, by $2\Delta t h^2 \left(u_j\right)^l$ and sum up over the nodes Ω_h^l

$$2h^2 \sum_{\Omega} \Delta t \rho \left(u_j\right)_{\bar{t}}^l u_j^l = h^2 \sum_{\Omega} \rho \left[\left(u_j^l\right)^2 - \left(u_j^{l-1}\right)^2 + \left(\Delta t\right)^2 \left(\left(u_j^l\right)_{\bar{t}} \right)^2 \right] \tag{13}$$

To transform the inertial terms we use the relations

$$\left(\overset{+i}{u} v_{x_i} + u v_{\bar{x}_i} \right) v = \left(u v \overset{-i}{v} \right)_{x_i} - u_{x_i} v^2$$

$$\Delta t h^2 \sum_{\Omega} \sum_{k=1}^{2} \left[\rho \overset{+k,-0}{u_k} \left(u_j\right)_{x_k} + \rho \overset{-0}{u_k} \left(u_j\right)_{\bar{x}_k} \right] u_j = \Delta t h^2 \sum_{\Omega} \sum_{k=1}^{2} \left[\left(\rho \overset{-0}{u_k} u_j \overset{-k}{u_j} \right)_{x_k} - \left(\rho \overset{-0}{u_k} \right)_{x_k} u_j^2 \right]$$

As $\sum_{\Omega} v_{x_i} = 0$ for any function which vanishes on the boundaries of computing region, then

$$\sum_{\Omega} \sum_{k=1}^{2} \left(\rho \overset{-0}{u_k} u_j \overset{-k}{u_j} \right)_{x_k} = 0.$$

Taking into account of (12), the second term is also equal to zero. Hence, as is shown in continuous case, convective term equals zero.

We extend the domain of u_j equating to zero outside of Ω_h^l. Applying summation by parts in the terms containing pressure and viscosity, we obtain

$$-2\Delta t h^2 \sum_{\Omega} f p_{\bar{x}_j} u_j = 2\Delta t h^2 \sum_{\Omega} p \left(f u_j \right)_{x_j} = 0$$

$$\Delta t h^2 \mu \sum_{\Omega_h^l} \sum_{k=1}^{2} \left(\frac{1}{3} \delta_j^k + 1 \right) \left[\left(f \left(u_j\right)_{x_k} \right)_{\bar{x}_k} + \left(f \left(u_j\right)_{\bar{x}_k} \right)_{x_k} \right] u_j$$

$$= -\Delta t h^2 \mu \sum_{\bar{\Omega}_h^l} \sum_{k=1}^{2} \left(\frac{1}{3} \delta_j^k + 1 \right) \left[\left(f \left(u_j\right)^2_{x_k} \right) + \left(f \left(u_j\right)^2_{\bar{x}_k} \right) \right],$$

$$\Delta t h^2 \mu \sum_{\Omega_h^l} \sum_{k=1}^{2} \left(1 - \frac{5}{3} \delta_j^k \right) \left[\left(f \left(u_{3-j}\right)_{\bar{x}_{3-k}} \right)_{x_k} + \left(f \left(u_{3-j}\right)_{x_{3-k}} \right)_{\bar{x}_k} \right] u_j$$

$$= -\Delta t h^2 \mu \sum_{\bar{\Omega}_h^l} \sum_{k=1}^{2} \left(1 - \frac{5}{3} \delta_j^k \right) \left[\left(f \left(u_{3-j}\right)_{\bar{x}_{3-k}} \right) \left(u_j\right)_{\bar{x}_k} + \left(f \left(u_{3-j}\right)_{x_{3-k}} \right) \left(u_j\right)_{x_k} \right] u_j. \tag{14}$$

Taking the summation equation (14) can be written as

$$h^2\sum_{\Omega_h^l}\rho\left[\left(u_j^l\right)^2-\left(u_j^{l-1}\right)^2+\left(\Delta t\right)^2\left(u_j^l\right)_{\bar{t}}^2\right]-2\Delta t h^2\sum_{\bar{\Omega}_h^l}p\left(fu_j\right)_{x_j}$$

$$+\Delta t h^2\mu\sum_{\bar{\Omega}_h^l}\sum_{k=1}^{2}\left\{\left(1+\tfrac{1}{3}\delta_j^k\right)\left[f\left(u_j\right)_{x_k}^2+f\left(u_j\right)_{\bar{x}_k}^2\right]\right.$$

$$\left.+\left(1-\tfrac{5}{3}\delta_j^k\right)f\left[\left(u_{3-j}\right)_{x_{3-k}}\left(u_j\right)_{x_k}+\left(u_{3-j}\right)_{x_{3-k}}\left(u_j\right)_{x_k}\right]\right\}$$

$$+2\Delta t h^2\sum_{\Omega_h^l}Ku_j^2=2\Delta t h^2\sum_{\Omega_h^l}\rho g_j u_j$$

Summing up these equalities over j we have

$$h^2\sum_{\Omega_h^l}\rho\sum_{j=1}^{2}\left(u_j^l\right)^2-h^2\sum_{\Omega_h^l}\rho\sum_{j=1}^{2}\left(u_j^{l-1}\right)^2+h\left(\Delta t\right)^2\sum_{\Omega_h^l}\rho\sum_{j=1}^{2}\left(u_j\right)_{\bar{t}}^2$$

$$-2\Delta t h^2\sum_{\bar{\Omega}_h^l}p\sum_{j=1}^{2}\left(fu_j\right)_{x_j}+\Delta t\mu L_h(v)+2\Delta t h^2\sum_{\Omega_h^l}K\sum_{j=1}^{2}u_j^2=2\Delta t h^2\sum_{\Omega_h^l}\rho g_j u_j, \qquad (15)$$

where

$$L_h(v)=h^2\sum_{\bar{\Omega}_h^l}\sum_{k=1}^{2}\sum_{j=1}^{2}f\left\{\left(\tfrac{1}{3}\delta_j^k+1\right)\left[\left(u_j\right)_{x_k}^2+\left(u_j\right)_{\bar{x}_k}^2\right]\right.$$

$$\left.+\left(1-\tfrac{5}{3}\delta_j^k\right)\left[\left(u_{3-j}\right)_{\bar{x}_{3-k}}\left(u_j\right)_{\bar{x}_k}+\left(u_{3-j}\right)_{x_{3-k}}\left(u_j\right)_{x_k}\right]\right\}$$

In the equality (15) terms containing pressure vanish due to the continuity equation.

We define net analogue of inner product and norm for net vector functions v_h, $w_h, \rho>0$ by formula

$$\left(v_h,w_h\right)_\rho=h^2\sum_{\bar{\Omega}_h}\rho\sum_{i=1}^{2}v_iw_i,\quad \left\|v_h\right\|_\rho^2=\left(v_h,v_h\right)_\rho$$

then we obtain from (15) the net analogue of equality (7)

$$\left\|v_h^l\right\|_\rho^2-\left\|v_h^{l-1}\right\|_\rho^2+\left(\Delta t\right)^2\left\|v_{\bar{t}}^l\right\|_\rho^2+\Delta t\mu L_h(v)+2\Delta t\bar{\alpha}\left\|v_h^l\right\|_{\bar{f}}^2=2\Delta t\left(G_h^l,v_h^l\right)_\rho \qquad (16)$$

The expression for $L_h(v)$ can be transformed as follows

$$L_h(v) = h^2 \sum_{\bar{\Omega}_h} f \left\{ \left(u_{1x_2} + u_{2x_1} \right)^2 + \left(u_{1x_2} + u_{2x_1} \right)^2 \right.$$
$$\left. + \frac{4}{3} \left(u_{1x_1}^2 + u_{2x_2}^2 + u_{1\bar{x}_1}^2 + u_{2\bar{x}_2}^2 - u_{1\bar{x}_1} u_{2\bar{x}_2} - u_{1x_1} u_{2x_2} \right) \right\} \tag{17}$$

As $L_h(v) > 0$, then according to the Cauchy-Schwartz inequality we obtain from (16) more rough inequality

$$\left\| v_h^l \right\|_\rho^2 - \left\| v_h^{l-1} \right\|_\rho^2 \le 2\Delta t \left\| G_h^l \right\|_\rho \left\| v_h^l \right\|_\rho ,$$

consequently

$$\left\| v_h^l \right\|_\rho \le \left\| v_h^{l-1} \right\|_\rho + 2\Delta t \left\| G_h^l \right\|_\rho \tag{18}$$

Summing up (16) over l from $l = 1$ to $l = m$ we have

$$\left\| v_h^m \right\|_\rho^2 - \left\| v_h^0 \right\|_\rho^2 + (\Delta t)^2 \sum_{l=1}^{m} \left\| v_{\bar{t}}^l \right\|_\rho^2 + \Delta t \mu \sum_{l=1}^{m} L_h(v) + 2\Delta t \bar{\alpha} \sum_{l=1}^{m} \left\| v_h^l \right\|_{\bar{f}}^2$$
$$= 2\Delta t \sum_{l=1}^{m} \left(G_h^l, v_h^l \right)_\rho \tag{19}$$

Summing up (16) on l from $l = 1$ to $l = m$, we get

$$\left\| v_h^m \right\|_\rho \le \left\| v_h^0 \right\|_\rho + 2\Delta t \sum_{l=1}^{m} \left\| G_h^l \right\|_\rho$$

From here it can be obtained the following inequality

$$\max_{1 \le l \le m} \left\| v_h^l \right\|_\rho \le \left\| v_h^0 \right\|_\rho + 2\Delta t \sum_{l=1}^{m} \left\| G_h^l \right\|_\rho \tag{20}$$

Let's estimate the right-hand side of the equality (19) by using the inequality (20)

$$\left|2\Delta t\sum_{l=1}^{m}\left(G_h^l, v_h^l\right)_\rho\right| \le 2\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho \left\|v_h^l\right\|_\rho \le 2\max_{1\le l\le m}\left\|v_h^l\right\|_\rho \Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho$$
$$\le 2\left(\left\|v_h^0\right\|_\rho + 2\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho\right)\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho$$
$$= 2\left\|v_h^0\right\|_\rho \Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho + 4\left(\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho\right)^2 \le \left\|v_h^0\right\|_\rho^2 + 5\left(\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho\right)^2.$$

Using this inequality, we obtain the difference analogue of the inequality (10)

$$\left\|v_h^m\right\|_\rho^2 + (\Delta t)^2\sum_{l=1}^{m}\left\|v_{\bar t}^l\right\|_\rho^2 + \Delta t\mu\sum_{l=1}^{m}L_h(v) + 2\Delta t\bar{\alpha}\sum_{l=1}^{m}\left\|v_h^l\right\|_{\bar f}^2$$
$$\le 2\left\|v_h^0\right\|_\rho^2 + 5\left(\Delta t\sum_{l=1}^{m}\left\|G_h^l\right\|_\rho\right)^2 \qquad (21)$$

The inequality (21) shows the stability of the constructed scheme.

CONCLUSION

On the basis of multi-fluid model under the assumption that the motion of discontinuous phase is absent the Rahmatullin equation is obtained which generalizes Navier-Stokes equations. Energy estimation for continuous problem is derived. Implicit finite-difference analogue of the Rahmatullin equation is derived and its stability is proved. The constructed scheme enables to explore complex flows containing porous rigid solid in it.

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